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Note on the scattering of long waves in a rotating system

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It is shown that the solution of the problem of scattering of long waves by a finite barrier in a rotating system may be obtained directly from the solution of the electromagnetic scattering problem for a finite strip. It is shown that the effect of rotation is to produce on both sides of the barrier, but away from the ends, a wave which is propagated without attenuation parallel to the barrier.

1. Introduction

The problem of scattering of long waves by a semi-infinite barrier in a rotating system was solved by Crease (1956) using the Wiener–Hopf method. Crease (1958) later used the same technique to consider the problem of the propagation of long waves into a semi-infinite channel in a rotating system. An alternative approach to the problem of scattering by a semi-infinite barrier has been suggested by Chambers (1964), who shows that the problem may be solved in a simple fashion without recourse to the complications of the Wiener–Hopf method. Chambers's approach is an adaptation of earlier work of his (Chambers 1954) on the Sommerfeld half-plane problem.

The present paper considers the problem of the scattering of a plane wave by a finite plane barrier in a rotating system. It is shown that a simple transformation enables the boundary-value problem to be reduced to one in electromagnetic diffraction theory and a formal solution thus obtained in terms of the solution of this latter problem. It is shown that in the vicinity of both sides of the barrier, but well away from both ends, there exist unattenuated waves of a type first observed by Crease (1956) for the semi-infinite barrier thus substantially confirming a conjecture of Crease's concerning the nature of the solution for the finite barrier.

The general approach of the present paper may be used to reduce any problem involving the scattering by plane parallel barriers in a rotating system to a Dirichlet problem for Helmholtz's equation. In particular the solution for the problem of scattering by two parallel semi-infinite scatterers may be deduced from the solution of a similar electromagnetic diffraction problem considered by Heins (1948).

2. Formulation and solution of the boundary-value problem

It will be assumed that the barrier is of length $2a$ and occupies the region $0 \leqslant x \leqslant 2a$ of the x-axis of a two-dimensional Cartesian system Oxy. The detailed formulation of this type of boundary-value problem has been given by Crease (1956) and from Crease's work it follows that the wave elevation ζ satisfies

$$\frac{\partial^2\zeta}{\partial x^2}+\frac{\partial^2\zeta}{\partial y^2}+k^2\zeta = 0, \tag{1}$$

where k is real. On the barrier ζ satisfies

$$\frac{\partial\zeta}{\partial y}-i\tanh\beta\frac{\partial\zeta}{\partial x} = 0, \tag{2}$$

where β is a real constant. [The constants k and β are defined in terms of the physical parameters of the problem in Crease (1956).] It will be assumed that there is a plane wave $\exp[ik(x\cos\theta_0+y\sin\theta_0)]$ incident on the barrier and hence apart from this term, ζ will consist of diffracted and reflected terms propagating outwards from the region occupied by the barrier. ζ is also required to be finite near the edge of the barrier.

If a solution ϕ of equation (1) is now defined so that $\zeta = D\phi$ where D is the operator $\partial/\partial y+i\tanh\beta\partial/\partial x$ then it follows from equations (1) and (2) that

$$\partial^2\phi/\partial x^2+k^2\cosh^2\beta\phi = 0, \quad y = 0, 0 \leqslant x \leqslant 2a. \tag{3}$$

Thus, on the barrier,

$$\phi = A\exp[ikx\cosh\beta]+B\exp[-ikx\cosh\beta], \tag{4}$$

where A and B are arbitrary constants. The boundary-value problem for ϕ satisfying equation (4), the condition of finiteness at the edges, and outgoing at infinity, possesses a unique solution for given values of A and B. There is, however, no known closed-form solution available for this problem except as an expansion in an infinite series involving Mathieu functions. The general boundary-value problem is, however, very closely related to the problem of the diffraction of an electromagnetic wave by a perfectly conducting strip, and a considerable amount of attention has been given to the solution of the latter problem for small and large values of the width of the strip (Millar 1958, 1960; Seshadri 1959).

The notation $\phi(\theta, x, y)$ will be used to denote the solution of the electromagnetic diffraction problem for an incident plane wave $\exp[ik(x\cos\theta+y\sin\theta)]$ and $\phi_d(\theta, x, y)$, the diffracted field, will be defined by

$$\phi(\theta, x, y) = \exp[ik(x\cos\theta+y\sin\theta)]+\phi_d(\theta, x, y).$$

The function ϕ vanishes on the strip and is such that ϕ_d is finite at the edges of the strip and represents an outgoing wave. In diffraction theory θ is of course real, but it can be verified by inspection of the various solutions that the methods of determining the function ϕ_d are still valid for imaginary values of θ and hence that the values of ϕ_d for such θ may be formally obtained from the solutions of the diffraction problem. It thus follows that a function satisfying equation (4) on the boundary is

$$\phi = -A\phi_d(i\beta, x, y)-B\phi_d(\pi-i\beta, x, y). \tag{5}$$

The substitution of the right-hand side of equation (5) into the equation $\zeta = D\phi$ will not yield the necessary incident plane wave term in ζ; this, however, is easily remedied by the addition of a suitable multiple of $\phi(\theta_0, x, y)$ to the right-hand side of equation (5). The final result is

$$\phi = \frac{\cos h\beta}{ik\sin(\theta_0 + i\beta)}\,\phi(\theta_0, x, y) - A\phi_d(i\beta, x, y) - B\phi_d(\pi - i\beta, x, y). \tag{6}$$

$D\phi$ with ϕ defined by equation (6) satisfies all the conditions imposed on ζ apart from those at the edges of the barrier, and it will now be shown that the imposition of the edge conditions enable A and B to be determined uniquely. If (r_1, θ_1) and (r_2, θ_2) denote polar co-ordinates with origins chosen at $x = 0$ and $2a$ respectively then it is well known from diffraction theory that, near $r_1 = 0$, the only singular contribution to grad $[\phi(\theta, x, y)]$ arises from a term of the form $E_1(\theta)\, r_1^{\frac{1}{2}} \sin \frac{1}{2}\theta_1$ where E_1 depends on the angle of incidence. Similarly, near $r_2 = 0$ the singular contribution to grad ϕ arises from a term $E_2(\theta)\, r_2^{\frac{1}{2}} \sin \frac{1}{2}\theta_2$. Thus, in order that ζ be finite at the edges, we have that

$$\frac{\cosh\beta}{ik\sin(\theta_0 + i\beta)}\,E_1(\theta_0) - AE_1(i\beta) - BE_1(\pi - i\beta) = 0, \tag{7}$$

$$\frac{\cosh\beta}{ik\sin(\theta_0 + i\beta)}\,E_2(\theta_0) - AE_2(i\beta) - BE_2(\pi - i\beta) = 0. \tag{8}$$

Hence
$$A = \frac{\cosh\beta}{ik\sin(\theta_0 + i\beta)}\,\frac{E_2(\theta_0)\,E_1(\pi - i\beta) - E_1(\theta_0)\,E_2(\pi - i\beta)}{E_2(i\beta)\,E_1(\pi - i\beta) - E_1(i\beta)\,E_2(\pi - i\beta)}, \tag{9}$$

$$B = \frac{\cosh\beta}{ik\sin(\theta_0 + i\beta)}\,\frac{E_2(\theta_0)\,E_1(i\beta) - E_1(\theta_0)\,E_2(i\beta)}{E_2(\pi - i\beta)\,E_1(i\beta) - E_1(\pi - i\beta)\,E_2(i\beta)}. \tag{10}$$

The functions E_1 and E_2 are not known in a simple closed form but for the case of large ka, which is likely to be the case of most practical interest, approximate expressions for E_1 and E_2 may be deduced from results obtained by Seshadri (1959). Seshadri has obtained approximate expressions for the Fourier transform of $\phi(\theta, x, y)$ and hence the edge behaviour of ϕ can be deduced from an examination of the asymptotic expansion of its Fourier transform. In fact it follows immediately from equations (3.34) and (3.35) of Seshadri's paper that, taking notational differences into account, apart from terms independent of θ,

$$E_1(\theta) = \sqrt{2}\cos\tfrac{1}{2}\theta - \frac{\exp i[2ka - \frac{1}{4}\pi]}{4\pi(2ka)^{\frac{3}{2}}}\exp[2ika\cos\theta]\tan\tfrac{1}{2}\theta\sec\tfrac{1}{2}\theta + O\left(\frac{1}{(ka)^2}\right), \tag{11}$$

$$E_2(\theta) = \sqrt{2}\exp[2ika\cos\theta]\sin\tfrac{1}{2}\theta - \frac{\exp i[2ka - \frac{1}{4}\pi]}{4\pi(2ka)^{\frac{3}{2}}}\cot\tfrac{1}{2}\theta\,\mathrm{cosec}\,\tfrac{1}{2}\theta + O\left(\frac{1}{(ka)^2}\right). \tag{12}$$

3. Discussion of results

The analysis of § 2 has enabled a formal solution to be obtained for ζ in terms of the solution ϕ of the electromagnetic problem. The solution of this latter problem is not known in a closed form, but for large values of ka a considerable amount of information is available concerning ϕ, but even the approximate expressions for ϕ are somewhat complicated. It is, however, possible to deduce from the above analysis without much elaborate calculation certain features of physical interest concerning the behaviour of the wave elevation ζ. For the case of an infinite barrier Crease (1956, 1958) observed that there exists, in the region near the barrier and above it, a wave travelling parallel to the barrier without attenuation. It is therefore of interest to examine whether a similar situation exists for the case of a finite barrier and Crease has in fact conjectured that in this case a certain amount of energy may be trapped in the form of a wave progressing round the barrier in a clockwise direction.

Before considering the effect of a large, but finite, barrier we shall show how the existence of the progressive wave near the infinite barrier may be deduced very simply from the above analysis. It should be noted that the solution for the infinite barrier is not necessarily the limit as $a \to \infty$ of that for the finite barrier and in this particular case the two solutions are different. This discrepancy is due to the fact that the present solution, which is a steady-state one, applies after an infinite time and thus we are comparing the solutions of the two problems: (i) time infinite followed by a infinite, and (ii) a infinite followed by time infinite, and these two problems do not necessarily have the same solution. A similar type of discrepancy occurs in the problem of electromagnetic diffraction by two parallel strips (Jones 1951). Formally, however, the results for the infinite strip may be deduced from these for the finite strip by introducing a slight dissipation into the system, i.e. by assuming that k has a positive imaginary part. This then gives

$$A = \frac{\cosh\beta}{ik\sin(\theta_0+i\beta)}\cos\tfrac{1}{2}\theta_0\,\mathrm{sech}\,\tfrac{1}{2}\beta \quad (B=0). \tag{13}$$

Alternatively these results could have been obtained immediately by adapting the analysis of § 2 to cover the case of a semi-infinite barrier. In this case the term involving B in equation (4) would violate the condition of outgoing waves at infinity, thus $B = 0$ and the expression for A then follows from equation (7) and the known form of $E_1(\theta_0)$.

It follows immediately from an inspection of the form of ϕ_d near the barrier, or by an intuitive extension of the results of geometrical optics that, apart from terms which vanish as the inverse root of the distance from the edge near the barrier

$$\phi_d(i\beta,x,y) \sim -\begin{cases}\exp[ikx\cosh\beta+ky\sinh\beta] & \text{for } \quad y<0,\\ \exp[ikx\cosh\beta-ky\sinh\beta] & \text{for } \quad y>0.\end{cases} \tag{14}$$

The operator D annihilates the plane-wave term in ϕ for $y < 0$ but for $y > 0$ it follows that the dominant term in the wave height away from the edge is given by

$$\frac{4i\cosh\beta\cos\frac{1}{2}\theta_0\sinh\frac{1}{2}\beta}{\sin(\theta_0+i\beta)}\exp\{ik(x\cosh\beta+y\sin i\beta)\}, \quad \text{for} \quad y>0. \tag{15}$$

The expression (15) agrees with expression (28) of Crease (1958) if it is noted that there is a slight algebraic slip in deriving Crease's expression (28) from his expression (26).

The above analysis may now be modified to investigate the behaviour of ζ near a large, but finite, barrier. Equation (14) still holds but we also have that, under the same conditions as equation (14),

$$\phi_d(\pi - i\beta, x, y) \sim -\begin{Bmatrix} \exp[-ikx\cosh\beta + ky\sinh\beta] & \text{for} & y < 0, \\ \exp[-ikx\cosh\beta - ky\sinh\beta] & \text{for} & y > 0. \end{Bmatrix} \tag{16}$$

It now follows immediately from equations (14) and (16) that, in the region just above the barrier and well away from both ends, the dominant term in the wave height is

$$-2k\sinh\beta A\exp[ik(x\cosh\beta + y\sin i\beta)]. \tag{17}$$

The corresponding result in the region just below the barrier and well away from the ends is

$$2k\sinh\beta B\exp[-ik(x\cosh\beta + y\sin i\beta)]. \tag{18}$$

Expressions (17) and (18) substantially confirm Crease's conjecture. It should also be noted that in the region just below the barrier, there will be an additional plane wave present due to the reflexion of the incident wave at the barrier.

The analysis of § 2 may be extended in an obvious fashion to reduce the problem of scattering by any number of plane parallel barriers to a Dirichlet problem for Helmholtz's equation. In certain cases this latter problem can be identified with a problem in electromagnetic diffraction and a formal solution obtained. One example of such a problem is the scattering of long waves by two parallel semi-infinite barriers—this problem was considered by Crease (1958). The solution of this problem can be obtained formally from the solution of a corresponding electromagnetic diffraction problem solved by Heins (1948). In fact the dominant behaviour of the wave elevation near the barriers may be deduced in an elementary fashion from Heins's work.

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REFERENCES

Chambers, L. G. 1954 *Proc. Edinb. Math. Soc.* **10**, 92.
Chambers, L. G. 1964 *Proc. Edinb. Math. Soc.* (in the Press).
Crease, J. 1956 *J. Fluid Mech.* **1**, 86.
Crease, J. 1958 *J. Fluid Mech.* **4**, 306.
Heins, A. E. 1948 *Quart. J. Math.* **6**, 157.
Jones, D. S. 1951 *Proc. Camb. Phil. Soc.* **48**, 118.
Millar, R. F. 1958 *Proc. Camb. Phil. Soc.* **54**, 479.
Millar, R. F. 1960 *Canad. J. Phys.* **38**, 38.
Seshadri, S. R. 1959 *Proc. Nat. Inst. Sci., India*, A, **25**, 301.